Theorem 4: Another definition of e.

Previously, we defined e to be the number such that e^x has slope 1 at $x=0$. Now we show

$$e = \lim_{x \to 0} (1+x)^{1/x} = \lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n \quad \text{(shown by Bernoulli)}$$

Proof: Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$.

Then $1 = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\ln(1+h)}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \ln(1+h) = \lim_{h \to 0} \ln((1+h)^{1/h})$

$= \ln\left(\lim_{h \to 0} (1+h)^{1/h}\right)$. Therefore $\lim_{h \to 0} (1+h)^{1/h} = e$.

3.9: Inverse Trig Fcns and Derivatives

$\sin^{-1} x = \arcsin x$ has range $[0, \pi]$

$\cos^{-1} x = \arccos x$ has range $[-\pi/2, \pi/2)$

$\tan^{-1} x = \arctan x$ has range $(-\pi/2, \pi/2)$

$\cot^{-1} x = \operatorname{arccot} x$ has range $(0, \pi)$

$\sec^{-1} x = \operatorname{arcsec} x$ has range $[0, \pi/2) \cup (\pi/2, \pi]$

$\csc^{-1} x = \operatorname{arccsc} x$ has range $[-\pi/2, 0) \cup (0, \pi/2]$

Ex: (1) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1-x^2}}$

Ex: (2) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1 + \tan^2(\tan^{-1} x)} = \frac{1}{1+x^2}$

Ex: (3) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{\sec(\sec^{-1}x)\tan(\sec^{-1}x)} = \pm\frac{1}{x\sqrt{\sec^2(\sec^{-1}x)-1}} = \pm\frac{1}{x\sqrt{x^2-1}}$

$$= \begin{cases} \frac{1}{x\sqrt{x^2-1}} & x > 1 \\ -\frac{1}{x\sqrt{x^2-1}} & x < -1 \end{cases} = \frac{1}{|x|\sqrt{x^2-1}}$$

Finally, $\cos^{-1}x = \pi/2 - \sin^{-1}x$

$\cot^{-1}x = \pi/2 - \tan^{-1}x$

$\csc^{-1}x = \pi/2 - \sec^{-1}x$

So that

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{d}{dx}(\sin^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

Similarly $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

$$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$$